

NON-EXISTENCE OF UNBOUNDED FATOU COMPONENTS OF A MEROMORPHIC FUNCTION

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Abstract

This paper is devoted to establish sufficient conditions under which a transcendental meromorphic function has no unbounded Fatou components and to extend some results for entire functions to meromorphic function. Actually, we shall mainly discuss non-existence of unbounded wandering domains of a meromorphic function. The case for a composition of finitely many meromorphic function with at least one of them being transcendental can be also investigated in the argument of this paper.

Keywords and Phases. Fatou set, Julia set

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1. INTRODUCTION AND MAIN RESULTS

Let \mathcal{M} be the family of all functions meromorphic in the complex plane \mathbb{C} possibly outside at most countable set, for example, a composition of finitely many transcendental meromorphic functions is in \mathcal{M} . Here we mean a function meromorphic in \mathbb{C} with only one essential singular point at ∞ by a transcendental meromorphic function. We shall study iterations of element in \mathcal{M} .

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We denote the n th iteration of $f(z) \in \mathcal{M}$ by $f^n(z) = f(f^{n-1}(z))$, $n = 1, 2, \dots$. Then $f^n(z)$ is well defined for all $z \in \mathbb{C}$ outside a (possible) countable set

$$E(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(E(f)),$$

here $E(f)$ is the set of all essential singular points of $f(z)$. Define the Fatou set $F(f)$ of $f(z)$ as

$$F(f) = \{z \in \bar{\mathbb{C}} : \{f^n(z)\} \text{ is well defined}$$

and normal in a neighborhood of $z\}$

and $J(f) = \bar{\mathbb{C}} \setminus F(f)$ is the Julia set of $f(z)$. $F(f)$ is open and $J(f)$ is closed, non-empty and perfect. It is well-known that both $F(f)$ and $J(f)$ are completely invariant under $f(z)$, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$. And $F(f^n) = F(f)$ and $J(f^n) = J(f)$ for any positive integer n . We shall consider components of the Fatou set $F(f)$ and hence let U be a connected component of $F(f)$. Since $F(f)$ is completely invariant under f , $f^n(U)$ is contained in $F(f)$ and connected, so there exists a Fatou component U_n such that $f^n(U) \subseteq U_n$. If for some $n \geq 1$, $f^n(U) \subseteq U$, that is, $U_n = U$, then U is called a periodic component of $F(f)$ and such the smallest integer n is the period of periodic component U . In particular, a periodic component of period one is also called invariant. If for some n , U_n is periodic, but U is not periodic, then U is called pre-periodic; A periodic component U of period p can be of the following five types: (i) attracting domain when U contains a point a such that $f^p(a) = a$ and $|(f^p)'(a)| < 1$ and $f^{np}|_U \rightarrow a$ as $n \rightarrow \infty$; (ii) parabolic domain when there exists a point $a \in \partial U$ such that $f^p(a) = a$ and $(f^p)'(a) = e^{2\pi i\alpha}$ for $\alpha \in \mathbb{Q}$ and $f^{np}|_U \rightarrow a$ as $n \rightarrow \infty$; (iii) Baker domain when $f^{np}|_U \rightarrow a \in \partial U \cup \{\infty\}$

as $n \rightarrow \infty$ and $f^p(z)$ is not defined at $z = a$; (iv) Siegel disk when U is simply connected and contains a point a such that $f^p(a) = a$ and $\phi \circ f^p \circ \phi^{-1}(z) = e^{2\pi i \alpha} z$ for some real irrational number α and a conformal mapping ϕ of U onto the unit disk with $\phi(a) = 0$; (v) Herman ring when U is doubly connected and $\phi \circ f^p \circ \phi^{-1}(z) = e^{2\pi i \alpha} z$ for some real irrational number α and a conformal mapping ϕ of U onto $\{1 < |z| < r\}$. U is called wandering if it is neither periodic nor preperiodic, that is, $U_n \cap U_m = \emptyset$ for all $n \neq m$. For the basic knowledge of dynamics of a meromorphic function, the reader is referred to [5] and the book [13].

If for a function $f \in \mathcal{M}$, $f^{-2}(E(f))$ contains at least three distinct points, then

$$J(f) = \overline{\bigcup_{n=1}^{\infty} f^{-n}(E(f))},$$

and in any case, what we should mention is that for every $n \geq 1$, $f^n(z)$ is analytic on $F(f)$. In particular, this result holds for a composition of finitely many meromorphic functions.

Our study in this paper relies on the Nevanlinna theory of value distribution. To the end, let us recall some basic concepts and notations in the theory. Let $f(z)$ be a meromorphic function in \mathbb{C} . Define

$$m(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ is the number of poles of $f(z)$ in the disk $\{|z| \leq t\}$, and

$$T(r, f) = m(r, f) + N(r, f)$$

which is known as the Nevanlinna characteristic function of $f(z)$. The quantity $\delta(\infty, f)$ is the Nevanlinna deficiency of f at ∞ , defined by the

following formula

$$\delta(\infty, f) = \liminf_{r \rightarrow \infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)}.$$

(See [6]). The growth order and lower order of $f(z)$ are defined respectively by

$$\lambda(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\mu(f) = \liminf_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}.$$

In this paper, we take into account the question, raised by I. N. Baker in 1984, of whether every component of $F(f)$ of a transcendental entire function $f(z)$ is bounded if its growth is sufficiently small. Baker [3] shown by an example that the order $1/2$ and minimal type is the best possible growth condition in terms of order. Following I. N. Baker's question, a number of papers gave some sufficient conditions which confirm Baker's question for the case of entire functions.

Zheng [15] made a discussion of non-existence of unbounded Fatou components of a meromorphic function and actually the method in [15] is available in proving the following

Theorem 1.1. *Let $f(z)$ be a function in \mathcal{M} . If we have*

$$(1) \quad \limsup_{r \rightarrow +\infty} \frac{L(r, f)}{r} = +\infty,$$

where $L(r, f) = \min\{|f(z)| : |z| = r\}$, then the Fatou set, $F(f)$, of f has no unbounded preperiodic or periodic components.

In particular, f has no Baker domains.

Theorem 1.1 confirms that an entire function whose growth does not exceed order $1/2$ and minimal type has no unbounded preperiodic or periodic components, whereas the result for the case of order less

than $1/2$ was proved in several papers, see [10] and [2]. In view of a well-known result that (1) is satisfied for a transcendental meromorphic function with lower order $\mu(f) < 1/2$ and $\delta(\infty, f) > 1 - \cos(\mu(f)\pi)$, Theorem 1.1 also confirms that such a meromorphic function has no unbounded preperiodic or periodic components. And it is described by an example in Zheng [15] that the condition (1) is sharpen. For a composition $g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z)$ of finitely many transcendental meromorphic functions $f_j(z) (j = 1, 2, \dots, m; m \geq 1)$, from the method of [15] it follows that $F(g)$ has no unbounded periodic or preperiodic components if for each j , there exists a sequence of positive real numbers tending to infinity at which $L(r, f_j) > r$ and (1) holds for at least one f_{j_0} .

Therefore, the crucial point solving I. N. Baker's question is in discussion of non-existence of unbounded wandering domains of a meromorphic function. There are a series of results for the case of entire functions on which some assumption on order less than $1/2$ and the certain regularity of the growth are imposed. Let $f(z)$ be an transcendental entire function with order $< 1/2$. Then every component of $F(f)$ is bounded, provided that one of the following statements holds:

- (1) $\frac{\log M(2r, f)}{\log M(r, f)} \rightarrow c \geq 1$ as $r \rightarrow \infty$, (Stallard [11], 1993);
- (2) $\frac{\varphi'(x)}{\varphi(x)} \geq \frac{c}{x}$, for all sufficiently large x , where $\varphi(x) = \log M(e^x, f)$ and $c > 1$ (Anderson and Hinkkanen [2], 1998);
- (3) $\log M(r^m, f) \geq m^2 \log M(r, f)$ for each $m > 1$ and all sufficiently large r (Hua and Yang [8], 1999);
- (4) $\mu(f) > 0$ (Wang [12], 2001).

A straightforward calculation deduces that an entire function satisfying the Stallard assumption with $c > 1$ must be of lower order at least $\log c / \log 2$. However, an entire function with $0 < \mu \leq \lambda(f) < \infty$ must

satisfy the Hua and Yang's assumption for m with $\mu(f)m > \lambda(f)$. In fact, choosing $\varepsilon > 0$ with $(\mu - \varepsilon)m > \lambda + 2\varepsilon$, we have for sufficiently large $r > 0$

$$(2) \quad \log M(r^m, f) > (r^m)^{\mu-\varepsilon} > r^\varepsilon r^{\lambda+\varepsilon} \geq r^\varepsilon \log M(r, f).$$

What we should mention is that by modify a little the proof given in [8], Hua and Yang's assumption for sufficiently large m instead of each $m > 1$ suffices to confirm their result to be true.

Zheng and Wang [16] in 2004 proved the following

Theorem 1.2. *Let $f(z)$ be a transcendental entire function. If there exists a $d > 1$ such that for all sufficiently large $r > 0$ we can find a $\tilde{r} \in [r, r^d]$ satisfying*

$$(3) \quad \log L(\tilde{r}, f) \geq d \log M(r, f),$$

then every component of $F(f)$ is bounded.

In [16] they also made a discussion of the case of composition of a number of entire functions. In 2005, Hinkkanen [7] also gave a weaker condition than (3), that is, the coefficient "d" before $\log M(r, f)$ is replaced by " $d(1 - (\log r)^{-\delta})$ " with $\delta > 0$.

In this paper, in view of the Nevanlinna theory of a meromorphic function, we consider the case of a meromorphic function and our main result is the following.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function and such that for some $\alpha \in (0, 1)$ and $D > d > 1$ and all the sufficiently large r , there exists an $t \in (r, r^d)$ satisfying*

$$(4) \quad \log L(t, f) > \alpha T(r, f), j = 1, 2, \dots, m.$$

and

$$(5) \quad T(r^d, f) \geq DT(r, f).$$

Then $F(f)$ has no unbounded components.

Actually, the assumption in Theorem 1.3 is also a sufficient condition of existence of buried points of the Julia set of a meromorphic function with at least one pole and which is not the form $f(z) = a + (z - a)^{-p}e^{g(z)}$. For such a meromorphic function, $J(f) = \overline{\bigcup_{j=0}^{\infty} f^{-j}(\infty)}$ and from Theorem 1.3 ∞ is a buried point of $f(z)$ and therefore so are all prepoles.

As a consequence of Theorem 1.3, we have the following

Theorem 1.4. *Let $f(z)$ be a transcendental meromorphic function with*

$$\delta(\infty, f) > 1 - \cos(\pi\lambda(f))$$

and $\lambda(f) < 1/2$ and $\mu(f) > 0$. Then $F(f)$ has no unbounded components.

In particular, Wang's result can be deduced from Theorem .

2. THE PROOF OF THEOREMS

To prove Theorems, we need some preliminary results. First preliminary result will be established by using the hyperbolic metric and it has independent significance. To the end, let us recall some properties on the hyperbolic metric, see ([1], [4]), etc. An open set W in \mathbb{C} is called hyperbolic if $\mathbb{C} \setminus W$ contains at least two points (note ∞ has been kicked out of W). Let U be a hyperbolic domains in \mathbb{C} . $\lambda_U(z)$ is the density of the hyperbolic metric on U and $\rho_U(z_1, z_2)$ stands for the

hyperbolic distance between z_1 and z_2 in U , i.e.

$$\rho_U(z_1, z_2) = \inf_{\gamma \in U} \int_{\gamma} \lambda_U(z) |dz|,$$

where γ is a Jordan curve connecting z_1 and z_2 in U . For a hyperbolic open set W , the hyperbolic density $\lambda_W(z)$ of W is the hyperbolic density for each component of W . Then we convent that the hyperbolic distance between two points which are in disjoint components equals to ∞ and the hyperbolic distance of two points a and b in one component U equals to $\rho_W(a, b) = \rho_U(a, b)$. For a fixed point $a \notin W$, introduce a domain constant

$$C_W(a) = \inf\{|z - a| \lambda_W(z) : z \in W\}.$$

If U is simply-connected and $d(z, \partial U)$ is a euclidean distance between $z \in U$ and ∂U , then for any $z \in U$,

$$(6) \quad \frac{1}{2d(z, \partial U)} \leq \lambda_U(z) \leq \frac{2}{d(z, \partial U)}.$$

Let $f : U \rightarrow V$ be analytic, where both U and V are hyperbolic domains. By the principle of hyperbolic metric, we have

$$(7) \quad \rho_V(f(z_1), f(z_2)) \leq \rho_U(z_1, z_2), \text{ for } z_1, z_2 \in U.$$

In particular, if $U \subset V$, then $\lambda_V(z) \leq \lambda_U(z)$ for $z \in U$.

Lemma 2.1. *(cf. Zheng [13]) Let U be a hyperbolic domain and $f(z)$ a function such that each $f^n(z)$ is analytic in U and $\bigcup_{n=0}^{\infty} f^n(U) \subset W$. If for some fixed point $a \notin W$, $C_W(a) > 0$ and $f^n|_U \rightarrow \infty$, then for any compact subset K of U there exists a positive constant $M = M(K)$ such that*

$$(8) \quad M^{-1}|f^n(z)| \leq |f^n(w)| \leq M|f^n(z)| \text{ for } z, w \in K.$$

Proof. Under the assumption of Lemma 2.1, we obtain

$$(9) \quad \rho_W(z) \geq \frac{C_W(a)}{|z-a|} \geq \frac{C_W(a)}{|z|+|a|}.$$

It follows that

$$(10) \quad \begin{aligned} \rho_{f^n(U)}(f^n(z), f^n(w)) &\geq \rho_W(f^n(z), f^n(w)) \\ &\geq C_W(a) \left| \int_{|f^n(z)|}^{|f^n(w)|} \frac{dr}{r+|a|} \right| \\ &= C_W(a) \left| \log \frac{|f^n(z)|+|a|}{|f^n(w)|+|a|} \right|. \end{aligned}$$

Set $A = \max\{\lambda_U(z, w) : z, w \in K\}$. Clearly $A \in (0, +\infty)$. From (7), we have

$$(11) \quad \rho_{f^n(U)}(f^n(z), f^n(w)) \leq \rho_U(z, w) \leq A.$$

Therefore, combining (10) and (11) gives

$$(12) \quad |f^n(z)|+|a| \leq (|f^n(w)|+|a|)e^{A/C_W(a)}.$$

This immediately completes the proof of Lemma 2.1.

The following is Lemma of Zheng [15](also see Theorem 1.6.7 of [13]).

Lemma 2.2. *Let $f : U \rightarrow U$ map the hyperbolic domain $U \subset \mathbb{C}$ analytically without fixed points and without isolated boundary points into itself. If $f^n|_U \rightarrow \infty (n \rightarrow \infty)$, then for any compact subset K of U , we have (8) for some $M = M(K) > 0$.*

The following is a consequence of Lemma 2.1 and Lemma 2.2, which is of independent significance.

Theorem 2.1. *Let $f(z)$ be a function in \mathcal{M} . If $F(f)$ contains an unbounded component, then for any compact subset K of $F(f)$ with*

$f^n|_K \rightarrow \infty$ as $n \rightarrow \infty$, we have a positive constant $M = M(K)$ such that (8) holds.

Proof. Assume without any loss of generalities that K is contained in a component U of $F(f)$. If $J(f)$ has one unbounded component, then we can find a subset Γ of $J(f)$ such that $\mathbb{C} \setminus \Gamma$ is simply-connected. Then in view of Lemma 2.1 we shall get $M = M(K)$ such that (8) holds by noting that $\bigcup_{n=0}^{\infty} f^n(U) \subset W = \mathbb{C} \setminus \Gamma$.

Now assume that $J(f)$ only has bounded components and thus $F(f)$ has only one unbounded component denoted by V . If $\bigcup_{n=0}^{\infty} f^n(U)$ does not intersect V , then in view of the fact that V has only bounded boundary components we can choose a path Γ in V tending to ∞ such that $\bigcup_{n=0}^{\infty} f^n(U) \subset W = \mathbb{C} \setminus \Gamma$. Thus as we did above, the result of Theorem 2.1 follows.

Let us consider the case when $U \subseteq \bigcup_{n=0}^{\infty} f^{-n}(V)$. If V is preperiodic or periodic, then an application of Lemma 2.2 yields the desired result of Theorem 2.1; If V is wandering, then for some $m > 1$, $\bigcup_{n=m}^{\infty} f^n(U)$ does not intersect V and therefore we can prove Theorem 2.1 in this case.

The second preliminary result comes from the Poisson formula.

Lemma 2.3. *Let $f(z)$ be meromorphic on $\{|z| \leq 3R\}$. Then there exists a $r \in (R, 2R)$ such that on $|z| = r$, we have*

$$(13) \quad \log^+ |f(z)| \leq KT(3R, f).$$

where $K(\leq 24)$ is a universal constant, that is, it is independent of R, r and f .

Proof. Set $D = \{|z| \leq \frac{5}{2}R\}$. We denote by $G_D(\zeta, z)$ the Green function of D , that is,

$$G_D(\zeta, z) = \log \left| \frac{(2.5R)^2 - \bar{z}\zeta}{2.5R(\zeta - z)} \right|, \quad z, \zeta \in D.$$

A simple calculation implies that

$$G_D(\zeta, z) \leq \log \frac{5R}{|\zeta - z|}$$

and for $\zeta = 2.5Re^{i\theta}$ and $r = |z| \leq 2R$,

$$\frac{\partial}{\partial \bar{n}} G_D(\zeta, z) ds = \operatorname{Re} \frac{2.5Re^{i\theta} + z}{2.5Re^{i\theta} - z} d\theta \leq \frac{2.5R + r}{2.5R - r} d\theta \leq 9d\theta.$$

In view of the Poisson formula, we have

$$\begin{aligned} \log |f(z)| &= \frac{1}{2\pi} \int_{\partial D} \log |f(\zeta)| \frac{\partial}{\partial \bar{n}} G_D(\zeta, z) ds \\ &\quad - \sum_{a_n \in D} G_D(a_n, z) + \sum_{b_n \in D} G_D(b_n, z) \\ &\leq 9m(2.5R, f) + \sum_{b_n \in D} \log \frac{5R}{|b_n - z|}, \end{aligned}$$

where a_n is a zero and b_n a pole of $f(z)$ in D counted according to their multiplicities. According to the definition of $N(r, f)$, we have

$$\begin{aligned} n(2.5R, f) &\leq \left(\log \frac{6}{5} \right)^{-1} \int_{2.5R}^{3R} \frac{n(t, f)}{t} dt \\ &\leq 6N(3R, f). \end{aligned}$$

From the Boutroux-Cartan Theorem it follows that

$$\prod_{n=1}^N |z - b_n| \geq \left(\frac{R}{2e} \right)^N, \quad N = n(2.5R, f),$$

for all $z \in \mathbb{C}$ outside at most N disks (γ) the total sum of whose diameters does not exceed $R/2$. Therefore there exists a $r \in [R, 2R]$ such that $\{|z| = r\} \cap (\gamma) = \emptyset$ and then on the circle $|z| = r$, we have

$$\log^+ |f(z)| \leq 9m(2.5R, f) + N \log 10e < 24T(3R, f).$$

Thus we complete the proof of Lemma 2.3.

Proof Of Theorem 1.3. For $\alpha > 0$, there exists a natural number k such that $D^{k-1}\alpha \geq 1$. Set $h = d^k$. In view of (4) and (5), for all $r \geq R_0$, we have a $t \in (r^{d^{k-1}}, r^h)$ such that

$$\begin{aligned} \log L(t, f) &\geq \alpha T(r^{d^{k-1}}, f) \geq \alpha D^{k-1} T(r, f) \\ (14) \quad &\geq T(r, f), \quad \text{on } |z| = t. \end{aligned}$$

From Lemma 2.3, we have

$$(15) \quad \log |f(z)| \leq KT(3r, f), \quad \text{for } |z| \leq 2r,$$

where K is a positive constant independent of f and r .

Take a positive integer m such that $D^{(m-1)k-1} > Kd^{mk} = Kh^m$. Suppose that f has an unbounded Fatou component, say U . Assume that U intersects $|z| = R_0$, otherwise we magnify R_0 . Take a point z_0 in $U \cap \{|z| = R_0\}$. Draw a curve $\gamma \in U$ from z_0 to $U \cap \{|z| = R_0^H\}$, $H = h^m$ such that $\gamma \subset \{|z| = R_0^H\}$ except the end point of γ .

Then there exists a $z_1 \in \gamma \cap \{R_0 \leq |z| \leq 2R_0\}$ such that $\log |f(z_1)| \leq KT(3R_0, f)$. And there exists a $r_1 \in (R_0^{h^{m-1}}, R_0^H)$ such that

$$\begin{aligned} \log L(r_1, f) &\geq T(R_0^{h^{m-1}}, f) = T(R_0^{d^{(m-1)k}}, f) \\ (16) \quad &\geq D^{(m-1)k-1} T(R_0^d, f) > Kh^m T(3R_0, f), \end{aligned}$$

on $|z| = r_1$. Set $R_1 = \exp(KT(3R_0, f_1))$. Then

$$(17) \quad f_1(\gamma) \cap \{|z| < R_1\} \neq \emptyset \text{ and } f_1(\gamma) \cap \{|z| > R_1^H\} \neq \emptyset.$$

By the same argument as above, we have a $z_2 \in f(\gamma) \cap \{R_1 \leq |z| \leq 2R_1\}$ such that $\log |f(z_2)| \leq KT(3R_1, f)$ and a $r_2 \in (R_1^{h^{m-1}}, R_1^H)$ such that

$$\log L(r_2, f) \geq h^m KT(3R_1, f), \quad \text{on } |z| = r_2.$$

Set $R_2 = \exp(KT(3R_1, f_2))$. Then since the circle $\{|z| = r_2\}$ intersects $f_1(\gamma)$, we have

$$(18) \quad f^2(\gamma) \cap \{|z| < R_2\} \neq \emptyset \text{ and } f^2(\gamma) \cap \{|z| > R_2^H\} \neq \emptyset.$$

Define $R_n = \exp(KT(3R_{n-1}, f))$ inductively. Then for each $n > 0$ we always have

$$f^n(\gamma) \cap \{|z| < R_n\} \neq \emptyset$$

and

$$f^n(\gamma) \cap \{|z| \geq R_n^H\} \neq \emptyset.$$

Thus there is two points $z_n, w_n \in \gamma$ such that

$$(19) \quad |f^n(z_n)| > R_n^H > |f^n(w_n)|^H.$$

Combining (19) and Theorem 2.1 gives

$$(20) \quad |f^n(w_n)|^H < |f^n(z_n)| \leq M|f^n(w_n)|.$$

This is impossible as $n \rightarrow \infty$, because a and e^{2A} are constants but $H > 1$ and $|f^n(z_n)| \rightarrow +\infty$ as $n \rightarrow +\infty$.

This completes the proof of Theorem 1.3.

To prove Theorem 1.4, we need the following result, which was proved by Gol'dberg and Sokolovskaya [9].

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function with $\delta(\infty, f) > 1 - \cos(\pi\lambda(f))$ and $\lambda(f) < 1/2$. Then*

$$\underline{\log \text{dens}} E > 0,$$

where $E = \{r > 0 : \log L(r, f) > \alpha T(r, f)\}$ for some positive α .

In fact Lemma 2.4 asserts that for sufficiently large $r > 0$, we can find a $t \in [r, r^d]$ for some $d > 1$ with

$$\log L(t, f) > \alpha T(r, f).$$

For a function $f(z)$ with $0 < \mu(f) \leq \lambda(f) < +\infty$, we easily see that

$$\lim_{r \rightarrow \infty} \frac{T(r^d, f)}{T(r, f)} = \infty$$

for d with $d\mu(f) > \lambda(f)$.

Therefore Theorem 1.4 follows immediately from Theorem 1.3.

3. CONCLUSION

By means of a careful calculation, indeed we can prove the following result: a transcendental meromorphic function $f(z)$ has no unbounded components of its Fatou set if for some $1 < d < D$ and all sufficiently large r there exists a $t \in [r, r^d]$ such that

$$\log L(t, f) > DT(r, f).$$

The argument of this paper is also available in establishing the corresponding results for a composition of finitely many meromorphic functions at least one of which is transcendental.

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